

# ON KÄHLER METRISABILITY FOR COMPLEX PROJECTIVE SURFACES

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**ABSTRACT.** We derive necessary conditions for a complex projective structure on a complex surface to arise via the Levi-Civita connection of a (pseudo-)Kähler metric. Furthermore we show that the (pseudo-)Kähler metrics defined on some domain in the projective plane which are compatible with the standard complex projective structure are in one-to-one correspondence with the hermitian forms on  $\mathbb{C}^3$  whose rank is at least two. This is achieved by prolonging the relevant finite-type first order linear differential system to closed form. Along the way we derive the complex projective Weyl and Liouville curvature using the language of Cartan geometries.

## 1. INTRODUCTION

Recall that an equivalence class of affine torsion-free connections on the tangent bundle of a smooth manifold  $N$  is called a (real) projective structure [8, 27, 28]. Two connections  $\nabla$  and  $\nabla'$  are *projectively equivalent* if they share the same unparametrised geodesics. This condition is equivalent to  $\nabla$  and  $\nabla'$  inducing the same parallel transport on the projectivised tangent bundle  $\mathbb{P}TN$ .

It is a natural task to (locally) characterise the projective structures arising via the Levi-Civita connection of a (pseudo-)Riemannian metric. R. Liouville [19] made that crucial observation that the Riemannian metrics on a surface whose Levi-Civita connection belongs to a given projective class precisely correspond to nondegenerate solutions of a certain projectively invariant finite-type linear system of partial differential equations. In [3] Bryant, Eastwood and Dunajski used Liouville's observation to solve the two-dimensional version of the Riemannian metrisability problem. Liouville's result generalises to higher dimensions [22] and the corresponding finite-type differential system was prolonged to closed form in [10, 22]. Several necessary conditions for Riemann metrisability of a projective structure in dimensions larger than two were given in [24]. See also [6, 12] for the role of Einstein metrics in projective geometry.

Now let  $M$  be a complex manifold of complex dimension  $d > 1$  with integrable almost complex structure map  $J$ . Two affine torsion-free connections  $\nabla$  and  $\nabla'$  on  $TM$  which preserve  $J$  are called *complex projectively equivalent* if they induce

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the same parallel transport on the complex projectivised tangent bundle  $\mathbb{P}T^{1,0}M$ . Complex projective geometry was introduced by Otsuki and Tashiro [25, 26]. Background on the history of complex projective geometry and its recently discovered connection to Hamiltonian 2-forms (see [1] and references therein) may be found in [20].

In the complex setting it is natural to study the *Kähler metrisability problem*, i.e. try to (locally) characterise the complex projective structures which arise via the Levi-Civita connection of a (pseudo-)Kähler metric. Similar to the real case, the Kähler metrics whose Levi-Civita connection belongs to a given complex projective class precisely correspond to nondegenerate solutions of a certain complex projectively invariant finite-type linear system of partial differential equations [20, Theorem 5].

In this note we prolong the relevant differential system to closed form in the surface case. In doing so we obtain necessary conditions for Kähler metrisability of a complex projective structure  $[\nabla]$  on a complex surface and show in particular that the generic complex projective structure is not Kähler metrisable. Furthermore we show that the space of Kähler metrics compatible with a given complex projective structure is algebraically constrained by the complex projective Weyl curvature of  $[\nabla]$ . We also show that the (pseudo-)Kähler metrics defined on some domain in  $\mathbb{CP}^2$  which are compatible with the standard complex projective structure are in one-to-one correspondence with the hermitian forms on  $\mathbb{C}^3$  whose rank is at least two. A result whose real counterpart is a well-known classical fact.

Computations are carried out using the parabolic Cartan geometry of a complex projective surface. Formulas for the complex projective Liouville - and Weyl curvature are also given.

This note has concerned itself with the complex 2-dimensional case, but there are obvious higher dimensional generalisations that can be treated with the same techniques.

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## 2. COMPLEX PROJECTIVE SURFACES

**2.1. Definitions.** Let  $M$  be a complex 2-manifold with integrable almost complex structure map  $J$  and  $\nabla$  an affine torsion-free connection on  $TM$ . We call  $\nabla$  *complex-linear* if  $\nabla J = 0$ . An *h-planar curve* or *generalised geodesic* for  $\nabla$  is a smoothly immersed curve  $\gamma \subset M$  with the property that the 2-plane spanned by  $\dot{\gamma}$  and  $J\dot{\gamma}$  is parallel along  $\gamma$ , i.e.  $\gamma$  satisfies the reparametrisation invariant condition

$$(1) \quad \nabla_{\dot{\gamma}} \dot{\gamma} \wedge \dot{\gamma} \wedge J\dot{\gamma} = 0.$$

We call two complex linear torsion-free connections  $\nabla$  and  $\nabla'$  on  $M$  *complex projectively equivalent* or *h-projectively equivalent*, if they have the same generalised geodesics. An equivalence class of complex projectively equivalent connections is called a *complex projective structure* or *h-projective structure* and will be denoted

by  $[\nabla]$ <sup>1</sup>. A complex 2-manifold equipped with a complex projective structure will be called a *complex projective surface*.

Extending  $\nabla$  to the complexified tangent bundle  $T^{\mathbb{C}}M \rightarrow M$ , it follows with the complex linearity of  $\nabla$  that for every local holomorphic coordinate system  $z = (z^i) : U \rightarrow \mathbb{C}^2$  on  $M$  there exist unique complex-valued functions  $\Gamma_{jk}^i$  on  $U$ , so that

$$\nabla_{\partial_{z^j}} \partial_{z^k} = \Gamma_{jk}^i \partial_{z^i}.$$

We call the functions  $\Gamma_{jk}^i$  the *complex Christoffel symbols* of  $\nabla$ . Tashiro showed [26] that two torsion-free complex linear connections  $\nabla$  and  $\nabla'$  on  $M$  are complex projectively equivalent if and only if there exists a  $(1,0)$ -form  $\beta \in \Gamma(T^{1,0}M^*)$  so that

$$(2) \quad \nabla'_Z W - \nabla_Z W = \beta(Z)W + \beta(W)Z$$

for all  $(1,0)$  vector fields  $Z, W \in \Gamma(T^{1,0}M)$ . In analogy to the real case one can use (2) to show that  $\nabla$  and  $\nabla'$  are complex projectively equivalent if and only if they induce the same parallel transport on the complex projectivised tangent bundle  $\mathbb{P}T^{1,0}M$ .

Writing  $\Gamma_{jk}^i$  and  $\hat{\Gamma}_{jk}^i$  for the complex Christoffel symbols of  $\nabla$  and  $\nabla'$  with respect to some holomorphic coordinates  $z = (z^i)$  and  $\beta = \beta_i dz^i$ , equation (2) translates to

$$(3) \quad \hat{\Gamma}_{jk}^i = \Gamma_{jk}^i + \delta_j^i \beta_k + \delta_k^i \beta_j.$$

Note that formally equation (3) is identical to the equation relating two real projectively equivalent connections on a real manifold. In particular, similarly to the real case (see [8, 27]), the functions

$$(4) \quad \Pi_{jk}^i = \Gamma_{jk}^i - \frac{1}{3} (\Gamma_{lj}^i \delta_k^l + \Gamma_{lk}^i \delta_j^l)$$

are complex projectively invariant in the sense that they only depend on the coordinates  $z$ . Moreover locally  $[\nabla]$  can be recovered from the functions  $\Pi_{jk}^i$  and two torsion-free complex linear connections are complex projectively equivalent if and only if they give rise to the same functions  $\Pi_{jk}^i$  in every holomorphic coordinate system.

A complex projective structure  $[\nabla]$  is called *holomorphic* if the functions  $\Pi_{jk}^i$  are holomorphic in every holomorphic coordinate system. Gunning [13] obtained relations on characteristic classes of complex manifolds carrying holomorphic projective structures. The condition on a manifold to carry a holomorphic projective structure is particularly restrictive in the case of compact complex surfaces. See also the beautiful twistorial interpretation of holomorphic projective surfaces by Hitchin [14] and Remark 3.

**2.2. Cartan geometry.** Similar to the real case, a complex projective structure admits a description in terms of a *Cartan geometry* which may be constructed using Cartan's method of equivalence. See [7] for a modern account on Cartan geometries and the appendix of [5] for background on Cartan's method of equivalence. We will restrict to the construction in the complex two-dimensional case. The general case may be found in [23] and [2] where a slightly different perspective is taken.

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<sup>1</sup>It is worth pointing out that the  $h$  in  $h$ -projective originally stood for *holomorphic*. Unfortunately,  $h$ -projective structures are not holomorphic in general. This is why we will henceforth use the terms complex projective equivalence and complex projective structure.

Let  $\mathrm{SL}(3, \mathbb{C})$  act on  $\mathbb{CP}^2$  from the left in the obvious way and let  $P$  denote the stabiliser subgroup of the element  $[1, 0, 0]^t \in \mathbb{CP}^2$ . Here we think of both  $\mathrm{SL}(3, \mathbb{C})$  and  $P$  as real Lie groups. The following theorem is a straightforward translation to the 2-dimensional complex case of Cartan's solution to the equivalence problem for real projective structures [8] (see also [16]).

**Theorem 1.** *Suppose  $(M, J, [\nabla])$  is a complex projective surface. Then there exists a (real) Cartan geometry  $(\pi : B \rightarrow M, \theta)$  of type  $(\mathrm{SL}(3, \mathbb{C}), P)$  such that for every local holomorphic coordinate system  $z = (z^i) : U \rightarrow \mathbb{C}^2$ , there exists a unique section  $\sigma_z : U \rightarrow B$  satisfying*

$$(5) \quad (\sigma_z)^* \theta = \begin{pmatrix} 0 & \phi_1^0 & \phi_2^0 \\ \phi_0^1 & \phi_1^1 & \phi_2^1 \\ \phi_0^2 & \phi_1^2 & \phi_2^2 \end{pmatrix}$$

where

$$\phi_0^i = dz^i, \quad \text{and} \quad \phi_j^i = \Pi_{jk}^i dz^k, \quad \text{and} \quad \phi_i^0 = \Pi_{ik} dz^k,$$

with

$$\Pi_{ij} = \Pi_{il}^k \Pi_{jk}^l - \frac{\partial \Pi_{ij}^k}{\partial z^k}$$

and  $\Pi_{jk}^i$  denote the complex projective invariants with respect to  $z^i$  defined in (4).

*Remark 1.* Suppose  $\varphi : (M, J, [\nabla]) \rightarrow (M', J', [\nabla]')$  is a biholomorphism between complex projective surfaces identifying the complex projective structures, then there exists a diffeomorphism  $\hat{\varphi} : B \rightarrow B'$  which is a  $P$ -bundle map covering  $\varphi$  and which satisfies  $\hat{\varphi}^* \theta' = \theta$ . Conversely, every diffeomorphism  $\Phi : B \rightarrow B'$  that is a  $P$ -bundle map and satisfies  $\Phi^* \theta' = \theta$  is of the form  $\Phi = \hat{\varphi}$  for a unique biholomorphism  $\varphi : M \rightarrow M'$  identifying the complex projective structures.

It is worth explaining how the generalised geodesics of  $[\nabla]$  appear in the Cartan geometry  $(\pi : B \rightarrow M, \theta)$ . To this end let  $G \subset P \subset \mathrm{SL}(3, \mathbb{C})$  denote the closed subgroup of upper triangular matrices. The quotient  $B/G$  is the total space of a fibre bundle over  $M$  whose fibre  $P/G$  is diffeomorphic to  $\mathbb{CP}^1$ . In fact,  $B/G$  may be identified with the complex projectivised tangent bundle  $\mathbb{PT}^{1,0}M$  of  $(M, J)$ . Writing

$$\theta = \begin{pmatrix} \theta_0^0 & \theta_1^0 & \theta_2^0 \\ \theta_0^1 & \theta_1^1 & \theta_2^1 \\ \theta_0^2 & \theta_1^2 & \theta_2^2 \end{pmatrix},$$

Theorem 1 implies that the real codimension 4-subbundle of  $TB$  defined by  $\theta_0^2 = \theta_1^2 = 0$  descends to a real rank 2 subbundle  $E \subset T\mathbb{PT}^{1,0}M$ . Furthermore, a smooth immersed curve  $\phi \subset \mathbb{PT}^{1,0}M$  satisfying  $\dot{\phi} \subset E$  projects to  $M$  to become a generalised geodesic of  $[\nabla]$ . Conversely, the lift  $\phi$  to  $\mathbb{PT}^{1,0}M$  of every generalised geodesic  $\gamma \subset M$  satisfies  $\dot{\phi} \subset E$ .

*Example 1.* Let  $B = \mathrm{SL}(3, \mathbb{C})$  and  $\theta = g^{-1}dg$  the Maurer-Cartan form of  $\mathrm{SL}(3, \mathbb{C})$ . Setting  $M = B/P \simeq \mathbb{CP}^2$  and  $\pi : \mathrm{SL}(3, \mathbb{C}) \rightarrow \mathbb{CP}^2$  the natural quotient projection, one obtains a complex projective structure on  $\mathbb{CP}^2$  whose generalised geodesics are the smooth immersed curves  $\gamma \subset \mathbb{CP}^1$  where  $\mathbb{CP}^1 \subset \mathbb{CP}^2$  is any linearly embedded projective line. This is precisely the complex projective structure associated to the Levi-Civita connection of the Fubini-Study metric on  $\mathbb{CP}^2$ . This example satisfies  $d\theta + \theta \wedge \theta = 0$  and is hence called *flat*.

Let  $(\pi : B \rightarrow M, \theta)$  be the Cartan geometry of a complex projective structure  $(J, [\nabla])$  on a simply-connected surface  $M$  whose Cartan connection satisfies  $d\theta + \theta \wedge \theta = 0$ . Then there exists a smooth map  $\Phi : B \rightarrow \mathrm{SL}(3, \mathbb{C})$  satisfying  $\theta = \Phi^{-1}d\Phi$  and consequently a local biholomorphism  $\varphi : M \rightarrow \mathbb{CP}^2$  identifying the projective structure on  $M$  with the standard flat structure on  $\mathbb{CP}^2$ .

**2.3. Bianchi-identities.** Theorem 1 implies that the curvature form  $\Theta = d\theta + \theta \wedge \theta$  satisfies

$$(6) \quad \Theta = d\theta + \theta \wedge \theta = \begin{pmatrix} 0 & \Theta_1^0 & \Theta_2^0 \\ 0 & \Theta_1^1 & \Theta_2^1 \\ 0 & \Theta_1^2 & \Theta_2^2 \end{pmatrix}$$

with

$$\Theta_i^0 = L_i \theta_0^1 \wedge \theta_0^2 + K_{il\bar{j}} \theta_0^l \wedge \bar{\theta}_0^{\bar{j}}, \quad \Theta_k^i = W_{kl\bar{j}}^i \theta_0^l \wedge \bar{\theta}_0^{\bar{j}}$$

for unique complex-valued functions  $L_i, K_{il\bar{j}}$ , and  $W_{kl\bar{j}}^i$  on  $B$  satisfying  $W_{li\bar{j}}^l = 0$ . Note that by construction, with respect to local holomorphic coordinates  $z = (z^i)$ , we obtain

$$(7) \quad (\sigma_z)^* W_{kl\bar{j}}^i = -\frac{\partial \Pi_{kl}^i}{\partial \bar{z}^j}.$$

Differentiation of the structure equations (6) gives

$$0 = d^2 \theta_0^i = W_{lk\bar{j}}^i \theta_0^l \wedge \theta_0^k \wedge \bar{\theta}_0^{\bar{j}}, \quad \text{and} \quad 0 = d^2 \theta_0^0 = K_{ik\bar{j}} \theta_0^i \wedge \theta_0^k \wedge \bar{\theta}_0^{\bar{j}}$$

which yields the algebraic Bianchi-identities

$$W_{lk\bar{j}}^i = W_{kl\bar{j}}^i, \quad \text{and} \quad K_{ik\bar{j}} = K_{ki\bar{j}}.$$

**2.3.1. Complex projective Weyl curvature.** The identities  $d^2 \theta_k^i = 0$  yield

$$\kappa_{kl\bar{j}}^i \wedge \theta_0^l \wedge \bar{\theta}_0^{\bar{j}} = 0$$

with

$$\kappa_{kl\bar{j}}^i = dW_{kl\bar{j}}^i + W_{kl\bar{j}}^i (\theta_0^0 + \bar{\theta}_0^0) + K_{kl\bar{j}} \theta_0^i - W_{ls\bar{j}}^i \theta_0^s - W_{ks\bar{j}}^i \theta_0^s + W_{kl\bar{j}}^s \theta_0^i - W_{kl\bar{s}}^i \bar{\theta}_0^{\bar{s}}$$

which implies that there exist complex-valued functions  $W_{kl\bar{j}\bar{s}}^i$  and  $W_{kl\bar{j}s}^i$  on  $B$  satisfying

$$W_{kl\bar{j}\bar{s}}^i = W_{lk\bar{j}\bar{s}}^i = W_{kl\bar{s}\bar{j}}^i, \quad W_{kl\bar{j}\bar{s}}^k = W_{kl\bar{j}s}^k = 0, \quad W_{kl\bar{j}s}^i = W_{lk\bar{j}s}^i$$

such that

$$(8) \quad dW_{kl\bar{j}}^i = (W_{kl\bar{j}s}^i + \delta_k^i K_{sl\bar{j}} + \delta_l^i K_{sk\bar{j}} - 3\delta_s^i K_{kl\bar{j}}) \theta_0^s + W_{kl\bar{j}\bar{s}}^i \bar{\theta}_0^{\bar{s}} + \varphi_{kl\bar{j}}^i$$

where

$$(9) \quad \varphi_{kl\bar{j}}^i = -W_{kl\bar{j}}^i (\theta_0^0 + \bar{\theta}_0^0) + W_{ls\bar{j}}^i \theta_0^s + W_{ks\bar{j}}^i \theta_0^s - W_{kl\bar{j}}^s \theta_0^i + W_{kl\bar{s}}^i \bar{\theta}_0^{\bar{s}}.$$

Let  $\mathrm{End}_0(TM, J)$  denote the bundle whose fibre at  $p \in M$  consists of the  $J$ -linear endomorphisms of  $T_p M$  which are complex-traceless. It follows with the structure equations (6,8,9) and straightforward computations, that there exists a unique  $(1,1)$ -form  $W$  on  $M$  with values in  $\mathrm{End}_0(TM, J)$  for which

$$W \left( \frac{\partial}{\partial z^l}, \frac{\partial}{\partial \bar{z}^j} \right) \frac{\partial}{\partial z^k} = (\sigma_z)^* W_{kl\bar{j}}^i \frac{\partial}{\partial z^i} = -\frac{\partial \Pi_{kl}^i}{\partial \bar{z}^j} \frac{\partial}{\partial z^i}$$

in every local holomorphic coordinate system  $z = (z^i)$  on  $M$ . Here, as usual, we extend tensor fields on  $M$  complex multilinearly to the complexified tangent bundle

of  $M$ . The bundle-valued 2-form  $W$  is called the *complex projective Weyl curvature* of  $[\nabla]$ .

*Remark 2.* In the case of real projective structures on surfaces, the projective Weyl curvature vanishes identically. Furthermore, in higher dimensions the complex projective Weyl tensor exists as well, but also contains (2,0) parts (see [26]).

We obtain:

**Proposition 1.** *A complex projective structure  $[\nabla]$  on a complex surface  $(M, J)$  is holomorphic if and only if the complex projective Weyl tensor of  $[\nabla]$  vanishes.*

2.3.2. *Complex projective Liouville curvature.* From  $d^2\theta_i^0 \wedge \overline{\theta_0^1} \wedge \overline{\theta_0^2} = 0$  one sees after a short computation that

$$(10) \quad dL_i = -4L_i\theta_0^0 + L_j\theta_i^j + L_{ij}\theta_0^j + L_{i\bar{j}}\overline{\theta_0^j}$$

for unique complex-valued functions  $L_{i\bar{j}}, L_{ij}$  on  $B$ . Using this last equation it is easy to check that the  $\pi$ -semibasic quantity

$$(11) \quad (L_1\theta_0^1 + L_2\theta_0^2) \otimes (\theta_0^1 \otimes \theta_0^2)$$

is invariant under the  $P$  right action and thus the  $\pi$ -pullback of a tensor field  $\lambda$  on  $M$  which is called the *complex projective Liouville curvature* (see the note of R. Liouville [18] for the construction of  $\lambda$  in the real case).

The differential Bianchi-identity (8) implies that if the functions  $W_{kl\bar{j}}^i$  vanish, then the functions  $K_{ik\bar{j}}$  must vanish as well. We have thus shown:

**Proposition 2.** *A complex projective structure  $[\nabla]$  on a complex surface  $(M, J)$  is flat if and only the complex projective Liouville and Weyl curvature vanish.*

*Remark 3.* In [17] Kobayashi and Ochiai classified compact complex surfaces carrying flat complex projective structures. More recently Dumitrescu [9] showed among other things that a holomorphic projective structure on a compact complex surface must be flat (see also the results by McKay about holomorphic Cartan geometries [21]).

2.3.3. *Further identities.* We also obtain

$$0 = d^2\theta_i^0 = \kappa_{ik\bar{j}} \wedge \overline{\theta_0^j} \wedge \theta_0^k$$

with<sup>2</sup>

$$\begin{aligned} \kappa_{ik\bar{j}} = & -dK_{ik\bar{j}} + \frac{1}{2}\varepsilon_{sk}L_{i\bar{j}}\theta_0^s - K_{ik\bar{j}}(2\theta_0^0 + \overline{\theta_0^0}) + K_{sk\bar{j}}\theta_i^s + K_{si\bar{j}}\theta_k^s - \\ & - W_{ik\bar{j}}^s\theta_s^0 + K_{ik\bar{s}}\overline{\theta_j^s}. \end{aligned}$$

It follows that there are complex-valued functions  $K_{ik\bar{j}l}$  and  $K_{kl\bar{i}j}$  on  $B$  satisfying

$$K_{ik\bar{j}l} = K_{ki\bar{j}l}, \quad \text{and} \quad K_{kl\bar{i}j} = K_{lk\bar{i}j} = K_{kl\bar{j}i}$$

such that

$$(12) \quad dK_{ik\bar{j}} = \left( K_{ik\bar{j}s} + \frac{1}{4}(\varepsilon_{sk}L_{i\bar{j}} + \varepsilon_{si}L_{k\bar{j}}) \right) \theta_0^s + K_{ik\bar{j}s}\overline{\theta_0^s} + \varphi_{ik\bar{j}}$$

where

$$\varphi_{ik\bar{j}} = -K_{ik\bar{j}}(2\theta_0^0 + \overline{\theta_0^0}) + K_{sk\bar{j}}\theta_i^s + K_{si\bar{j}}\theta_k^s - W_{ik\bar{j}}^s\theta_s^0 + K_{ik\bar{s}}\overline{\theta_j^s}.$$

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<sup>2</sup>We write  $\varepsilon_{ij}$  for the antisymmetric 2-by-2 matrix satisfying  $\varepsilon_{12} = 1$  and  $\varepsilon^{ij}$  for the inverse matrix.

**2.4. Complex geodesics.** Recall the rank 2 subbundle  $E \subset T\mathbb{P}T^{1,0}M$  defined by the equations  $\theta_1^2 = \theta_0^2 = 0$  on  $B$ . It is natural to ask when  $E$  is integrable. It follows with the structure equations (6) that

$$d\theta_0^2 = 0 \mod \theta_0^2, \theta_1^2$$

and

$$d\theta_1^2 = W_{11\bar{7}}^2 \theta_0^1 \wedge \overline{\theta_0^2} \mod \theta_0^2, \theta_1^2.$$

Consequently,  $E$  is integrable if and only if  $W_{11\bar{1}}^2 = W_{11\bar{2}}^2 = 0$ . Using the structure equations (8) and (9) it is easy to check that this implies that  $E$  is integrable if and only if  $[\nabla]$  is holomorphic. Suppose  $[\nabla]$  is holomorphic so that  $E$  is integrable. In analogy to real projective surfaces it follows that the leaves of the foliation defined by  $E$ , when projected down to  $M$ , are immersed complex curves  $Y \subset M$  for which  $\nabla_Y \dot{Y}$  is proportional to  $\dot{Y}$  for some (and hence any)  $\nabla \in [\nabla]$ . This last condition is easily seen (c.f. [23, Lemma 4.1]) to be equivalent to  $Y$  being a totally geodesic immersed complex curve with respect to  $([\nabla], J)$ . Conversely, the natural lift of such a curve  $Y \subset M$  to  $\mathbb{P}T^{1,0}M$  is a leaf of the foliation  $E$ . A totally geodesic immersed complex curve  $Y \subset M$  which is maximally extended is called a *complex geodesic*. We may summarise:

**Proposition 3.** *Let  $(M, J, [\nabla])$  be a complex projective surface. Then the following statements are equivalent:*

- (i)  *$[\nabla]$  is holomorphic;*
- (ii) *The complex projective Weyl tensor of  $[\nabla]$  vanishes;*
- (iii) *The rank 2 bundle  $E \rightarrow \mathbb{P}T^{1,0}M$  is Frobenius integrable;*
- (iv) *Every complex line  $L \subset T^{1,0}M$  is tangent to a unique complex geodesic.*

*Remark 4.* The standard flat complex projective structure on  $\mathbb{CP}^2$  is holomorphic and the complex geodesics are simply the linearly embedded projective lines  $\mathbb{CP}^1 \subset \mathbb{CP}^2$ .

### 3. KÄHLER METRISABILITY

In this section we will derive necessary conditions for a complex projective structure  $[\nabla]$  on a complex surface  $(M, J)$  to arise via the Levi-Civita connection of a (pseudo-)Kähler metric. Similarly to the real case first studied by R. Liouville [19], there exists a complex projectively invariant linear first order differential operator  $D_{[\nabla]}$  acting on hermitian  $(0,2)$  tensor fields on  $M$ , tensored with a (complex) density of weight  $-2/3$ , with the following property: The Levi-Civita connection of a (pseudo-)Kähler metric  $g$  on  $(M, J)$  is projectively equivalent to  $[\nabla]$  if and only if

$$(13) \quad D_{[\nabla]} \left( g \otimes (\det_{\mathbb{C}} g)^{-2/3} \right) = 0.$$

Also, similar to the real case, (13) is an equation of finite-type and can be prolonged to a closed system.

**3.1. The differential analysis.** On the level of the Cartan geometry  $(\pi : B \rightarrow M, \theta)$ , the characterisation of Kähler metrics compatible with  $[\nabla]$  given by (13) can be expressed as follows:

**Proposition 4.** *Suppose the Kähler metric  $g$  is compatible with  $[\nabla]$ . Then, writing  $\pi^*g = g_{i\bar{j}}\theta_0^i \circ \overline{\theta_0^j}$  and setting  $h_{i\bar{j}} = g_{i\bar{j}}(g_{1\bar{1}}g_{2\bar{2}} - |g_{1\bar{2}}|^2)^{-2/3}$ , we have*

$$(14) \quad dh_{i\bar{j}} = h_{i\bar{j}} \left( \theta_0^0 + \overline{\theta_0^0} \right) + h_{i\bar{s}} \overline{\theta_j^s} + h_{s\bar{j}} \theta_i^s + h_i \overline{\varepsilon_{sj} \theta_0^s} + \overline{h_j} \varepsilon_{si} \theta_0^s$$

for some complex-valued functions  $h_i$  on  $B$ . Conversely, suppose there exist complex-valued functions  $h_{i\bar{j}} = \overline{h_{j\bar{i}}}$  and  $h_i$  on  $B$  solving (14) and satisfying  $(h_{1\bar{1}}h_{2\bar{2}} - |h_{1\bar{2}}|^2) \neq 0$ , then the symmetric 2-form

$$h_{i\bar{j}} (h_{1\bar{1}}h_{2\bar{2}} - |h_{1\bar{2}}|^2)^{-2} \theta_0^i \circ \overline{\theta_0^j}$$

is the  $\pi$ -pullback of a  $[\nabla]$ -compatible Kähler metric on  $M$ .

*Proof.* Using Theorem 1, holomorphic normal coordinates for  $g$  and straightforward computations show that (14) is necessary. Conversely, suppose there exist complex-valued functions  $h_{i\bar{j}} = \overline{h_{j\bar{i}}}$  and  $h_i$  on  $B$  solving (14) for which

$$(h_{1\bar{1}}h_{2\bar{2}} - |h_{1\bar{2}}|^2) \neq 0.$$

Setting  $g_{i\bar{j}} = h_{i\bar{j}} (h_{1\bar{1}}h_{2\bar{2}} - |h_{1\bar{2}}|^2)^{-2}$  we get

$$(15) \quad dg_{i\bar{j}} = -g_{i\bar{j}} (\theta_0^0 + \overline{\theta_0^0}) + g_{i\bar{s}} \overline{\theta_j^s} + g_{s\bar{j}} \theta_i^s + g_{i\bar{s}} \overline{\theta_0^s} + g_{i\bar{j}s} \theta_0^s$$

with

$$g_{i\bar{j}s} = \frac{(h_{i\bar{j}}h_{l\bar{s}} + h_{i\bar{s}}h_{l\bar{j}})\varepsilon^{lk}h_k}{(h_{1\bar{1}}h_{2\bar{2}} - |h_{1\bar{2}}|^2)^3}, \quad \text{and} \quad g_{i\bar{j}k} = \frac{(h_{i\bar{j}}h_{k\bar{s}} + h_{k\bar{j}}h_{i\bar{s}})\overline{\varepsilon^{su}h_u}}{(h_{1\bar{1}}h_{2\bar{2}} - |h_{1\bar{2}}|^2)^3}.$$

Straightforward calculations show that the equations  $h_i = 0$  define a principal right  $\text{GL}(2, \mathbb{C})$ -subbundle  $F \subset B$  which is easily seen to be isomorphic to the complex linear frame bundle of  $(M, J)$ . On  $F$  equation (15) simplifies to

$$(16) \quad dg_{i\bar{j}} = g_{i\bar{s}} \overline{\theta_j^s} + g_{s\bar{j}} \theta_i^s$$

where  $\theta_j^i$  are connection forms on  $F$  belonging to a  $[\nabla]$ -compatible complex-linear torsion-free connection on  $TM$ . Using (16) it is easy to check that the symmetric 2-form

$$g_{i\bar{j}} \theta_0^i \circ \overline{\theta_0^j}$$

descends to a Kähler metric on  $M$  whose Levi-Civita connections forms are given by the  $\theta_j^i$ .  $\square$

3.1.1. *First prolongation.* Differentiating (14) yields

$$(17) \quad 0 = d^2 h_{i\bar{j}} = \varepsilon_{si} \overline{\eta_j} \wedge \theta_0^s + \overline{\varepsilon_{sj}} \eta_i \wedge \overline{\theta_0^s} - (h_{s\bar{j}} W_{i\bar{v}\bar{u}}^s + h_{i\bar{s}} \overline{W_{j\bar{u}\bar{v}}^s}) \overline{\theta_0^u} \wedge \theta_0^v$$

with

$$\eta_k = dh_k + h_k (\overline{\theta_0^0} - \theta_0^0) - h_j \theta_k^j + \overline{\varepsilon^{ij}} h_{k\bar{j}} \overline{\theta_0^i}.$$

This implies that we can write

$$(18) \quad \eta_i = a_{ij} \theta_0^j$$

for unique complex-valued functions  $a_{ij}$  on  $B$ . Equations (17) and (18) imply

$$(19) \quad \varepsilon_{ki} \overline{a_{jl}} - \overline{\varepsilon_{lj}} a_{ik} = \overline{h_{j\bar{s}}} W_{i\bar{k}\bar{l}}^s - h_{i\bar{s}} \overline{W_{j\bar{l}\bar{k}}^s}$$

Contracting this last equation with  $\overline{\varepsilon^{jl}} \varepsilon^{ik}$  implies that the function

$$h = -\frac{1}{2} \overline{\varepsilon^{ij}} a_{ij}$$



is real-valued. We get

$$a_{jl} = \varepsilon_{jl}h - \frac{1}{2}\overline{\varepsilon^{i\bar{u}}}h_{s\bar{i}}W_{jl\bar{u}}^s.$$

and thus

$$dh_i = h_i(\theta_0^0 - \overline{\theta_0^0}) + h_j\theta_i^j + h_{i\bar{s}}\overline{\varepsilon^{sl}}\theta_l^0 + \left(\varepsilon_{ij}h - \frac{1}{2}\overline{\varepsilon^{uv}}h_{s\bar{u}}W_{ij\bar{v}}^s\right)\theta_0^j.$$

Plugging the formula for  $a_{ij}$  back into (19) yields the integrability conditions

$$h_{s\bar{j}}W_{ik\bar{l}}^s - h_{i\bar{s}}\overline{W_{jl\bar{k}}^s} = \frac{1}{2}\overline{\varepsilon_{lj}}\varepsilon^{uv}h_{s\bar{u}}W_{ik\bar{v}}^s - \frac{1}{2}\varepsilon_{ki}\varepsilon^{uv}h_{u\bar{s}}\overline{W_{jl\bar{v}}^s}$$

which is equivalent to

$$(20) \quad \overline{h_{j\bar{s}}W_{ik\bar{l}}^s} + \overline{h_{l\bar{s}}W_{ik\bar{j}}^s} = h_{k\bar{s}}\overline{W_{jl\bar{i}}^s} + h_{i\bar{s}}\overline{W_{jl\bar{k}}^s}.$$

**Proposition 5.** *A necessary condition for a complex projective surface  $(M, J, [\nabla])$  to be Kähler metrisable is that the system (20) admits a nondegenerate solution  $\overline{h_{ij}} = h_{j\bar{i}}$ .*

*Remark 5.* Note that under suitable constant rank assumptions the system (20) defines a subbundle of the bundle over  $M$  whose sections are hermitian forms on  $(M, J)$ . For a generic projective structure  $[\nabla]$  this subbundle does have rank 0.

3.1.2. *Second prolongation.* We start by computing

$$0 = d^2h_i \wedge \theta_0^1 \wedge \theta_0^2 = -\left(h_{ij}\overline{\varepsilon^{jk}}L_k\right)\theta_0^1 \wedge \overline{\theta_0^1} \wedge \theta_0^2 \wedge \overline{\theta_0^2}$$

which is equivalent to

$$\begin{pmatrix} h_{1\bar{1}} & h_{1\bar{2}} \\ h_{2\bar{1}} & h_{2\bar{2}} \end{pmatrix} \cdot \begin{pmatrix} \overline{L_2} \\ -\overline{L_1} \end{pmatrix} = 0$$

which cannot have any solution with  $(h_{11}h_{22} - |h_{12}|^2) \neq 0$  unless  $L_1 = L_2 = 0$ . This shows:

**Theorem 2.** *A necessary condition for a complex projective surface  $(M, J, [\nabla])$  to be Kähler metrisable is that it is Liouville-flat, i.e. its complex projective Liouville curvature vanishes.*

Assuming henceforth  $L_1 = L_2 = 0$  we also get

$$(21) \quad 0 = d^2h_i = (\varepsilon_{ij}\eta + \varphi_{ij}) \wedge \theta_0^j$$

with

$$\eta = dh + 2h\text{Re}(\theta_0^0) + 2\varepsilon^{ij}\text{Re}(h_i\theta_j^0) - \frac{1}{2}\varepsilon^{kl}h_{k\bar{i}}\overline{\varepsilon^{ij}}K_{j\bar{s}\bar{l}}\theta_0^s$$

and

$$\varphi_{ij} = dr_{ij} + r_{ij}\overline{\theta_0^0} - r_{si}\theta_j^s - r_{sj}\theta_i^s - h_l W_{ij\bar{s}}^l \overline{\theta_0^s} + \frac{1}{2}\overline{\varepsilon^{uv}}(h_{i\bar{u}}\overline{K_{vs\bar{j}}} + h_{j\bar{u}}\overline{K_{vs\bar{i}}})\overline{\theta_0^s}$$

where

$$r_{ij} = -\frac{1}{2}\overline{\varepsilon^{uv}}h_{s\bar{u}}W_{ij\bar{v}}^s.$$

It follows with Cartan's lemma that there are functions  $a_{ijk} = a_{ikj}$  such that

$$\varepsilon_{ij}\eta + \varphi_{ij} = a_{ijk}\theta_0^k.$$

Since  $\varphi_{ij}$  is symmetric in  $i, j$ , this implies

$$\eta = \frac{1}{2} \varepsilon^{ji} a_{ijs} \theta_0^s.$$

Since  $h$  is real-valued, we must have

$$\varepsilon^{ji} a_{ijs} = \overline{\varepsilon^{uv}} \varepsilon^{kl} h_{k\bar{u}} K_{ls\bar{v}}.$$

Concluding, we get

$$dh = -2h \operatorname{Re}(\theta_0^0) + 2\varepsilon^{kl} \operatorname{Re}(h_l \theta_k^0) + \frac{1}{2} \overline{\varepsilon^{ij}} \varepsilon^{kl} \operatorname{Re}(h_{k\bar{i}} K_{ls\bar{j}} \theta_0^s).$$

This completes the prolongation procedure.

*Remark 6.* Note that further integrability conditions can be derived from (21), we won't write these out though.

Using Proposition 4 we obtain:

**Theorem 3.** *Let  $(M, J, [\nabla])$  be a complex projective surface with Cartan geometry  $(\pi : B \rightarrow M, \theta)$ . If  $U \subset B$  is a connected open set on which there exist functions  $h_{i\bar{j}} = \overline{h_{j\bar{i}}}$ ,  $h_i$  and  $h$  that satisfy the rank 9 linear system*

$$\begin{aligned} (22) \quad dh_{i\bar{j}} &= 2h_{i\bar{j}} \operatorname{Re}(\theta_0^0) + h_{i\bar{s}} \overline{\theta_j^s} + h_{s\bar{j}} \theta_i^s + h_i \overline{\varepsilon_{s\bar{j}} \theta_0^s} + \overline{h_j} \varepsilon_{si} \theta_0^s, \\ dh_k &= 2ih_k \operatorname{Im}(\theta_0^0) + h_l \theta_k^l + h_{k\bar{i}} \overline{\varepsilon^{ij} \theta_j^0} + \left( \varepsilon_{kl} h - \frac{1}{2} \overline{\varepsilon^{ij}} h_{s\bar{i}} W_{kl\bar{j}}^s \right) \theta_0^l, \\ dh &= -2h \operatorname{Re}(\theta_0^0) - 2\varepsilon^{lk} \operatorname{Re}(h_l \theta_k^0) + \frac{1}{2} \overline{\varepsilon^{ij}} \varepsilon^{kl} \operatorname{Re}(h_{k\bar{i}} K_{ls\bar{j}} \theta_0^s), \end{aligned}$$

and  $(h_{1\bar{1}} h_{2\bar{2}} - |h_{1\bar{2}}|^2) \neq 0$ , then the quadratic form

$$g = \frac{h_{i\bar{j}} \theta_0^i \circ \overline{\theta_0^j}}{(h_{1\bar{1}} h_{2\bar{2}} - |h_{1\bar{2}}|^2)^2}$$

is the  $\pi$ -pullback to  $U$  of a (pseudo-)Kähler metric on  $\pi(U) \subset M$  that is compatible with  $[\nabla]$ .

From this we get:

**Corollary 1.** *The Kähler metrics defined on some domain  $U \subset \mathbb{CP}^2$  which are compatible with the standard complex projective structure on  $\mathbb{CP}^2$  are in one-to-one correspondence with the hermitian forms on  $\mathbb{C}^3$  whose rank is at least two.*

*Proof.* Suppose the complex projective structure  $[\nabla]$  has vanishing complex projective Weyl and Liouville curvature. Then the differential system (22) may be written as

$$(23) \quad dH + \theta H + H\theta^* = 0$$

with

$$H = H^* = \begin{pmatrix} h & -\overline{h_2} & \overline{h_1} \\ -h_2 & -h_{22} & h_{21} \\ h_1 & h_{12} & -h_{11} \end{pmatrix}$$

where  $*$  denotes the conjugate transpose matrix. Recall that in the flat case  $\theta = g^{-1} dg$  for some smooth  $g : B \rightarrow \operatorname{SL}(3, \mathbb{C})$ , hence the solutions to (23) are

$$H = g^{-1} C (g^{-1})^*$$

where  $C = C^*$  is a constant hermitian matrix of rank at least two. The statement now follows immediately with Theorem 3.  $\square$

*Remark 7.* One can deduce from Corollary 1 that a Kähler metric  $g$  giving rise to flat complex projective structures must have constant holomorphic sectional curvature. A result first proved in [26] (in all dimensions).

*Remark 8.* One can also ask for existence of complex projective structures  $[\nabla]$  whose *degree of mobility* is greater than one, i.e. they admit several (non-proportional) compatible Kähler metrics. In [11] (see also [15]) it was shown that the only closed complex projective manifold with degree of mobility greater than two is  $\mathbb{CP}^n$  with the projective structure arising via the Fubini-Study metric.

## REFERENCES

- [1] V. APOSTOLOV, D. M. J. CALDERBANK, and P. GAUDUCHON, Hamiltonian 2-forms in Kähler geometry. I. General theory, *J. Differential Geom.* **73** (2006), 359–412. MR 2228318
- [2] S. ARMSTRONG, Projective holonomy. I. Principles and properties, *Ann. Global Anal. Geom.* **33** (2008), 47–69. MR 2369186
- [3] R. BRYANT, M. DUNAJSKI, and M. EASTWOOD, Metrisability of two-dimensional projective structures, *J. Differential Geom.* **83** (2009), 465–499. MR 2581355 Zbl 1196.53014
- [4] R. L. BRYANT, *Notes on projective surfaces*, private manuscript in progress.
- [5] R. L. BRYANT and P. A. GRIFFITHS, Characteristic cohomology of differential systems. II. Conservation laws for a class of parabolic equations, *Duke Math. J.* **78** (1995), 531–676. MR 1334205
- [6] A. ČAP, A. R. GOVER, and H. R. MACBETH, *Einstein metrics in projective geometry*, arXiv:1207.0128, 2012.
- [7] A. ČAP and J. SLOVÁK, *Parabolic geometries. I, Mathematical Surveys and Monographs* **154**, American Mathematical Society, Providence, RI, 2009, Background and general theory. MR 2532439
- [8] E. CARTAN, Sur les variétés à connexion projective, *Bull. Soc. Math. France* **52** (1924), 205–241. MR 1504846 Zbl 50.0500.02
- [9] S. DUMITRESCU, Connexions affines et projectives sur les surfaces complexes compactes, *Math. Z.* **264** (2010), 301–316. MR 2574978
- [10] M. EASTWOOD and V. MATVEEV, Metric connections in projective differential geometry, in *Symmetries and overdetermined systems of partial differential equations, IMA Vol. Math. Appl.* **144**, Springer, New York, 2008, pp. 339–350. MR 2384718
- [11] A. FEDOROVA, V. KIOSAK, V. S. MATVEEV, and S. ROSEMAN, The only Kähler manifold with degree of mobility at least 3 is  $(\mathbb{CP}(n), g_{\text{Fubini-Study}})$ , *Proc. Lond. Math. Soc. (3)* **105** (2012), 153–188. MR 2948791
- [12] A. R. GOVER and H. R. MACBETH, *Detecting Einstein geodesics: Einstein metrics in projective and conformal geometry*, arXiv:1212.6286, 2012.
- [13] R. C. GUNNING, *On uniformization of complex manifolds: the role of connections*, *Mathematical Notes* **22**, Princeton University Press, Princeton, N.J., 1978. MR 505691
- [14] N. J. HITCHIN, Complex manifolds and Einstein’s equations, in *Twistor geometry and nonlinear systems (Primorsko, 1980)*, *Lecture Notes in Math.* **970**, Springer, Berlin, 1982, pp. 73–99. MR 699802
- [15] K. KIOHARA and P. TOPALOV, On Liouville integrability of  $h$ -projectively equivalent Kähler metrics, *Proc. Amer. Math. Soc.* **139** (2011), 231–242. MR 2729086
- [16] S. KOBAYASHI and T. NAGANO, On projective connections, *J. Math. Mech.* **13** (1964), 215–235. MR 0159284
- [17] S. KOBAYASHI and T. OCHIAI, Holomorphic projective structures on compact complex surfaces, *Math. Ann.* **249** (1980), 75–94. MR 575449
- [18] R. LIOUVILLE, Sur une classe d’équations différentielles, parmi lesquelles, en particulier, toutes celles des lignes géodésiques se trouvent comprises., *Comptes rendus hebdomadaires des séances de l’Académie des sciences* **105** (1887), 1062–1064.

- [19] R. LIOUVILLE, Sur les invariants de certaines équations différentielles et sur leurs applications, *Journal de l'Ecole Polytechnique* **59** (1889), 7–76.
- [20] V. S. MATVEEV and S. ROSEMAN, Proof of the Yano-Obata conjecture for h-projective transformations, *J. Differential Geom.* **92** (2012), 221–261. MR 2998672
- [21] B. MCKAY, Characteristic forms of complex Cartan geometries, *Adv. Geom.* **11** (2011), 139–168. MR 2770434
- [22] J. MIKEŠ, Geodesic mappings of affine-connected and Riemannian spaces, *J. Math. Sci.* **78** (1996), 311–333, Geometry, 2. MR 1384327
- [23] R. MOLZON and K. P. MORTENSEN, The Schwarzian derivative for maps between manifolds with complex projective connections, *Trans. Amer. Math. Soc.* **348** (1996), 3015–3036. MR 1348154
- [24] P. NUROWSKI, Projective vs metric structures, *J. Geom. Phys.* **62** (2012), 657–674.
- [25] T. ŌTSUKI and Y. TASHIRO, On curves in Kaehlerian spaces, *Math. J. Okayama Univ.* **4** (1954), 57–78. MR 0066024
- [26] Y. TASHIRO, On a holomorphically projective correspondence in an almost complex space, *Math. J. Okayama Univ.* **6** (1957), 147–152. MR 0087181
- [27] T. Y. THOMAS, On the projective and equi-projective geometries of paths., *Proc. Nat. Acad. Sci.* **11** (1925), 199–203.
- [28] H. WEYL, Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung., *Göttingen Nachrichten* (1921), 99–112. Zbl 48.0844.04

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